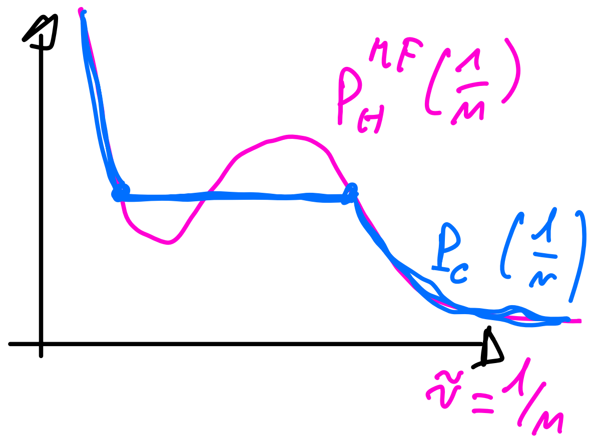
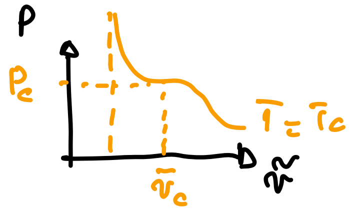
⇒ the "real" P is thus constant in the phase separated state





Universality

Let us look at the behaviour of the system close to P_c, T_c, μ_c.

$$P(\tilde{v}) = P(\tilde{v}_c) + \left.\frac{\partial P}{\partial \tilde{v}}\right|_{T_c} (\tilde{v}-\tilde{v}_c) + \frac{1}{2}\frac{\partial^2 P}{\partial \tilde{v}^2}(\tilde{v}-\tilde{v}_c)^2 + \frac{1}{6}\frac{\partial^3 P}{\partial \tilde{v}^3}(\tilde{v}-\tilde{v}_c)^3 + \dots \quad (1)$$

At T_c, V_c, $\frac{\partial P}{\partial \tilde{v}} = \frac{\partial^2 P}{\partial \tilde{v}^2} = 0$ Mean-field: $P = \frac{kT}{\tilde{v} - \frac{\Omega}{2}} - \frac{u}{2\tilde{v}^2}$

$$P'(\tilde{v}) = 0 \Leftrightarrow -\frac{kT_c}{(\tilde{v}_c - \frac{\Omega}{2})^2} + \frac{u}{\tilde{v}_c^3} = 0 \Leftrightarrow \frac{\tilde{v}_c^3}{u} = \frac{(\tilde{v}_c - \frac{\Omega}{2})^2}{kT_c} \quad (*)$$

$$P''(\tilde{v}) = 0 \Leftrightarrow \frac{2kT_c}{(\tilde{v}_c - \frac{\Omega}{2})^3} - \frac{3u}{\tilde{v}_c^4} \Leftrightarrow \frac{\tilde{v}_c^4}{3u} = \frac{(\tilde{v}_c - \frac{\Omega}{2})^3}{2kT_c} \quad (**)$$

$$\frac{(**)}{(*)} \Rightarrow \frac{\tilde{v}_c}{3} = \frac{(\tilde{v}_c - \frac{\Omega}{2})}{2} \Rightarrow \boxed{\tilde{v}_c = \frac{3\Omega}{2}}$$

$$(*) \Rightarrow \boxed{kT_c} = u \cdot \frac{(\tilde{v}_c - \frac{\Omega}{2})^2}{\tilde{v}_c^3} = u \cdot \frac{\Omega^2 8}{27\Omega^3} = \boxed{\frac{8u}{27\Omega}}$$

$$\boxed{P_c = \frac{2u}{27\Omega^2}}$$

⇒ Three quantities that depend on two parameters, Ω & u ⇒ 1 dimensionless combination

Universal quantity: $\frac{p_c \tilde{v}_c}{k_B T_c} = \frac{3}{8}$ $\Rightarrow$ parameter free!

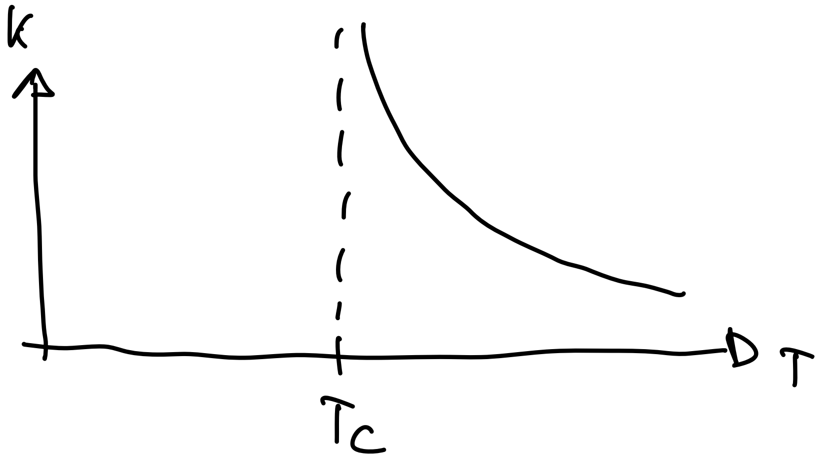
exp: Many systems leads to $\frac{p_c \tilde{v}_c}{k_B T_c} \in [0.28, 0.33]$

Not universal but close!

(Δ) $\Rightarrow$ At $T = T_c$, $p - p_c \propto (\tilde{v} - \tilde{v}_c)^3$ exp: $p - p_c \propto (\tilde{v} - \tilde{v}_c)^{\delta}$ with $\delta = 5.00$ for all simple gases. Universal!

Compressibility: $K = -\frac{1}{\tilde{v}} \frac{\partial \tilde{v}}{\partial p}$

$$p'(\tilde{v})\Big|_{v_c} = \frac{8a}{27 b^3} - \frac{kT}{b^2} = \frac{k(T_c - T)}{b^2} \Rightarrow K \propto \frac{1}{T - T_c} \xrightarrow[T \to \infty]{} \infty$$



exp: $K \sim \frac{1}{|T - T_c|^{1.24}}$

for all simple gases

Universal!

Two surprises: (1) There are universal observables, that are identical among quite different systems & models.

(2) They are not quantitatively predicted by mean-field.

Why? Because fluctuations are important. $\rho(\vec{r}) \simeq \rho_0$ is too simple an approximation. Learn more? 8.334!

4.4) The Ferromagnetic transition & the mean-field Ising model

The Ising model is a simplified model to account for the exchange interactions between electrons in a solid. Consider a lattice of L^d sites in d dimensions. At each site, we associate a value $s_i \in \{+1,-1\}$ (which correspond to spin 1/2 or -1/2) and consider the Hamiltonian:

$$H = -J\sum_{i\sim j} s_i s_j - \hat{h}\mu\sum_i s_i$$

where J is the coupling constant (the exchange energy), $\hat{h}$ is a magnetic field, μ is the magnetic moment of the electrons (For simplicity we denote $h = \mu\hat{B}$), and $\sum_{i\sim j}$ runs over all pairs of nearest neighbors.

* A configuration of the system is described by the values of $\{s_i\} \Rightarrow 2^N$ configurations, when $N = L^d$ is the number of spins.
* When $J=0$, the spins are non-interacting and the system can be treated as a collection of non-interacting two-level systems.
* H shows that the spins tend to align with h. This is called paramagnetism and the system's magnetization $M = \sum_{i=1}^{N} s_i$ tends to align with h.
* Here we want to understand the emergence of ferromagnetism, which describes systems such that $|\langle m\rangle| = \left|\left\langle \frac{1}{N}\sum_{i=1}^{N} s_i \right\rangle\right|$ remains non zero

as $N\to\infty$ in the absence of magnetic field. In such systems, the exchange interactions leads to an emergent magnetization.

* Ferromagnets correspond to $J>0$, which favors $\uparrow\uparrow$ & $\downarrow\downarrow$, while $J<0$ correspond to antiferromagnets, which favors $\uparrow\downarrow$ & $\downarrow\uparrow$.

Canonical ensemble:

* At $T=\infty$, $P(\{S_i\}) = \frac{1}{Z}$ and the 2^N configurations are equally likely so that there will be no net magnetization.

* At $T=0$, $P(\{S_i\})=0$ if all the spins are not aligned & $m=\pm 1$.

Q: What happens in between?

Partition function: Let's keep h finite for now.

$$Z = \sum_{\{S_i\}} e^{\beta\left[J\sum_{\langle ij\rangle} S_i S_j + h\sum_i S_i\right]}$$

so that $\langle M\rangle = \frac{\partial}{\partial(\beta h)} \ln Z \Big|_T \Rightarrow \boxed{\langle m\rangle\Big|_{h=0} = \frac{k_B T}{N}\frac{\partial}{\partial h}\ln Z\Big|_{h=0}}$

$\Rightarrow$ Can we compute Z?

$d=1$. Yes! Transfer matrix

$d=2$. Yes (at $h=0$)! Pain & suffering (Onsager, 1942)

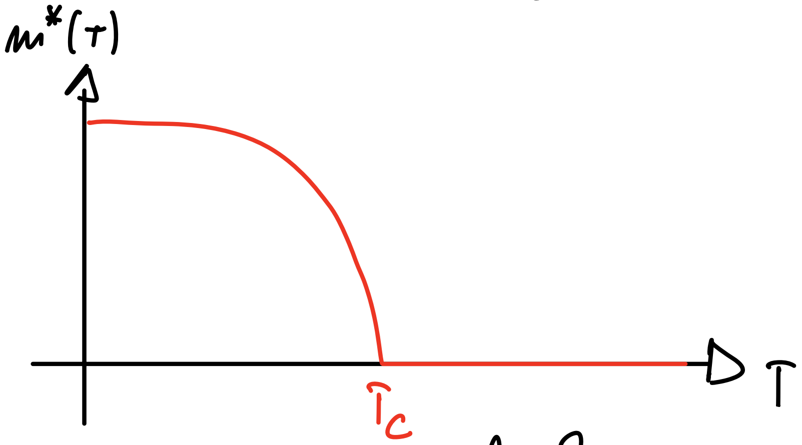
less ——————— : Wigner-Jordan transformation.

$d=3$. No, :-c, but recent advances due to extension of conformal field theory

& lots of numerics ⇒ we understand the physics.

Result:

As $L \to \infty$, $P(m) \to \frac{1}{2}\delta(m+m^*) + \frac{1}{2}\delta(m-m^*)$



Above the critical "Curie" temperature, $m \simeq 0$. Below $m = \pm m^*(T) \sim \mathcal{O}(1)$.

Can we understand this?

Mean-field theory:

$$H = -J \sum_{i \sim j} S_i S_j - h \sum_i S_i$$

Let us consider the contributions involving spin i: $H_i = -hS_i - \left(J \sum_{j \sim i} S_j\right) S_i$

$J \sum_{j \sim i} S_j$ can be seen as the effective magnetic field that the neighbours of i induce on spin i. If the system is homogeneous & the fluctuations are small,

$$\boxed{\sum_{j \in v(i)} S_j \simeq q m}$$

, where q is the number of neighbours of spin i ($2d$ on a square lattice in d dimension).

This leads to $H_i = -(h + qmJ) S_i$

This is like the Hamiltonian of a single spin in a field $h_{eff} = h + qmJ$

⑥

$\Rightarrow$ we can solve that problem and compute $\bar{m}(h_{eff}) \equiv \langle m \rangle_{h_{eff}}$

& make sure that the approximation is self-consistent, i.e.

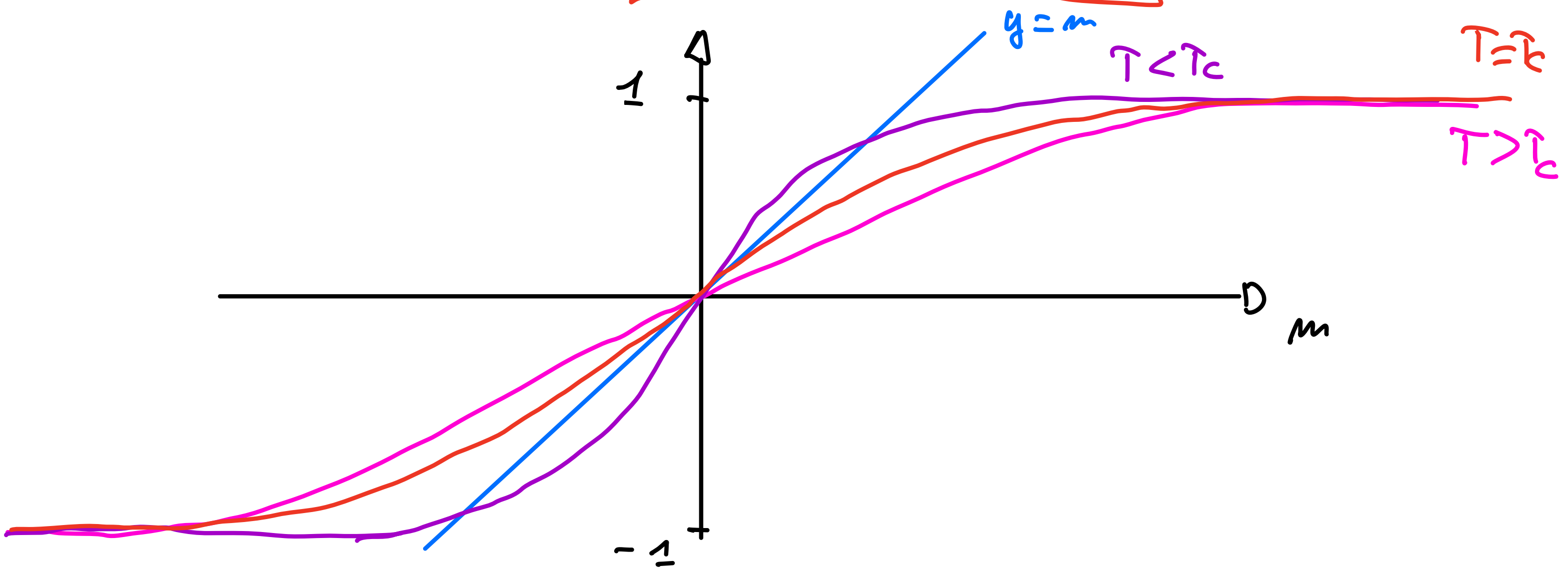
$$\bar{m}\left(\beta_{eff}(m)\right) = m.$$

Partition function $Z'_{eff} = e^{-\beta(h+qmJ)} + e^{\beta(h+qmJ)}$

$$\Rightarrow \bar{m}(B_{eff}) = \frac{\partial}{\partial(\beta h)} \log\left[e^{\beta(h+qmJ)} + e^{-\beta(h+qmJ)}\right]$$

$$\bar{m}(B_{eff}) = \tanh\left[\beta(h+qmJ)\right]$$

at $h=0$, we want $\boxed{m = \tanh[\beta q m J]}$



For $T > T_c$, only one solution, $m=0$.

$T < T_c$, three solutions, $m = \pm m_0$ & $m=0$.

Like in the liquid gas transition, when we had 3 solution $\tilde{v}^*_g$, $\tilde{v}^*_l$ & $\tilde{v}_c$, the middle one, here $m=0$, is not a local minimum of the Landau free energy.

Looking for T_c : small m expansion

$$\tanh(x) \underset{x\to 0}{\simeq} x \Rightarrow m = \beta q m J$$

$$T < T_c \iff \beta q J > 1 \iff \boxed{T < \frac{qJ}{k_B} = T_c}$$

Critical exponent

order parameter

Expanding $\tanh(x) = x - \frac{x^3}{3}$, we see that

$$T < T_c \Rightarrow m \simeq \beta q m J - \frac{(\beta q m J)^3}{3}$$

$$\iff (\beta - \beta_c) q J m = \frac{1}{3}\beta^3 q^3 J^3 m^3 \Rightarrow m \propto \pm (T_c - T_c)^\beta ; \beta = \frac{1}{2}$$

$(2d : \beta = \frac{1}{8})$

Magnetic field

$$m \simeq \beta h + m\frac{T_c}{T} - \frac{m^3}{3}\left(\frac{T_c}{T}\right)^3 \text{, using } T_c = \frac{qJ}{k_B}$$

$$T = T_c \Rightarrow m \propto h^{1/3}$$

$$T \neq T_c \Rightarrow m\left(\frac{T - T_c}{T}\right) = \beta h \Rightarrow \chi = \frac{\partial m}{\partial h}\Big|_{h=0} \sim \frac{1}{T - T_c}$$

Like for the Van der Waal's fluid!

$S_i = \pm 1 \rightarrow n_i = \frac{1+S_i}{2} \in [0,1] \Rightarrow$ lattice model for attractive fluid!

The world of phase transitions is full of such surprises!